Surface Integral & Stokes theorem

- surfaces
- surface area

We are familiar with curves in the plane. The study of surfaces in the space $\mathbb{R}^3$ is somehow parallel to it.

First, parametric surfaces.

A <u>parametric surface</u> is a continuous map from some set $E \subset \mathbb{R}^2$ to $\mathbb{R}^3$. The set E is quite flexible, usually it is an open set or assume the form $[a,b] \times [c,d]$. Thus write

$$\vec{r} : E \to \mathbb{R}^3, \quad \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \quad \text{or}$$
$$= x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}.$$

It is regular if the vectors

$$\frac{\partial \vec{r}}{\partial u} = (x_u, y_u, z_u) \quad \text{and}$$

$$\frac{\partial \vec{r}}{\partial v} = (x_v, y_v, z_v)$$

are linearly independent at every $(u,v) \in E$. Recall that two 3-vectors $\vec{a}, \vec{b}$ are linearly independent iff $\vec{a} \times \vec{b} \neq (0,0,0)$.
A parametric surface is regular if x, y, z are C^1-functions of u, v and

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \neq (0,0,0), \quad \text{or}$$

$$\left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| > 0 \quad \text{on } E.$$

There are 3 ways to define a (geometric) surface $\subset \mathbb{R}^3$. The first one is to use parametric surfaces.

A set S in $\mathbb{R}^3$ is called a <u>regular</u> <u>surface</u> if there is a regular parametric surface $\vec{r}: E \to \mathbb{R}^3$ such that it maps E 1-1 onto S.

The second approach is the <u>explicit</u> <u>surfaces</u>. Let $f(x,y)$ be a C^1-function over some $(x,y) \in G$, G open set $\subset \mathbb{R}^2$. Then the set

$$\{(x, y, f(x,y)) : (x,y) \in G\}$$

forms an explicit surface. (In fact, $\underline{(x,y) \mapsto (x,y,f(x,y))}$ can be viewed as a parametrization. We have

$$\frac{\partial \vec{r}}{\partial x} = (1, 0, f_x), \quad \frac{\partial \vec{r}}{\partial y} = (0, 1, f_y)$$

which is clearly linearly independent. Similar situation applies to

$$(x, g(x,z), z), \quad \text{and} \quad (h(y,z), y, z).$$

The third approach is the <u>implicit</u> <u>surfaces</u>. Very often, a surface is given by the level set of some fcn $\subset \mathbb{R}^3$. Consider the set

$$\{(x,y,z) : F(x,y,z) = c\}, \quad c \text{ constant}.$$

If $\frac{\partial F}{\partial z} \neq 0$ at some pt $\vec{r}_0 = (x_0, y_0, z_0)$ in this set, by the implicit function theorem, then exist some function $f(x,y)$ for

$(x,y) \in$ some open set G s.t.

$$F(x,y,f(x,y)) = c,$$

that's, the level set of $F=c$ is described as $z=f(x,y)$ near the pt $\vec{r}_0$. Similar situations hold when F_y or $F_z \neq 0$. Putting things together, if

$$\nabla F(\vec{r}_0) \neq (0,0,0),$$

then the level set $F=c$ is an explicit surface near $\vec{r}_0$.

$\boxed{\text{eg } 1}$. the sphere.

$$F(x,y,z) = x^2 + y^2 + z^2.$$

$$\nabla F = 2(x,y,z) \neq (0,0,0) \quad \text{if} \quad (x,y,z) \neq (0,0,0).$$

So, the level sets of $F=c$ define implicit surfaces. In fact $F(x,y,z) = a^2$ $(a>0)$ is the sphere of radius a.

Explicit form can be obtained by specifying a pt on it. Taking $\vec{r}_0 = (0,0,1)$,

$$f(x,y) = \sqrt{a^2-x^2-y^2}, \quad (x,y) \in D_1$$

describe the hemisphere (upper) containing $(0,0,1)$. For $(0,0,-1)$, it is

$$-\sqrt{a^2-x^2-y^2}.$$

For $(1,0,0)$, it is

$$\sqrt{a^2-y^2-z^2}, \quad (y,z) \in D_1.$$

Parametric description can be obtained using polar coordinates

$$(\theta,\varphi)\longmapsto(\cos\theta\sin\varphi,\ \sin\theta\sin\varphi,\ \cos\varphi)=\vec{r}(\theta,\varphi)$$

$$\frac{\partial\vec{r}}{\partial\theta}=(-\sin\theta\sin\varphi,\ \cos\theta\sin\varphi,\ 0)$$

$$\frac{\partial\vec{r}}{\partial\varphi}=(\cos\theta\cos\varphi,\ \sin\theta\cos\varphi,\ -\sin\varphi)$$

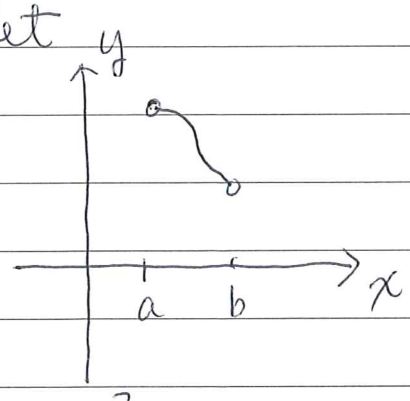
When $-\sin\varphi\neq 0$, that is $\varphi\neq 0,\pi$, $\frac{\partial\vec{r}}{\partial\theta}$ and $\frac{\partial\vec{r}}{\partial\varphi}$ are linearly independent.

When $\varphi=0$ or π, $\frac{\partial\vec{r}}{\partial\theta}=(0,0,0)$, so no good. We conclude that $\vec{r}$ is a regular parametrization of the sphere on $[0,2\pi]\times(0,\pi)$.

$\square$

A class of surfaces commonly encountered are <u>surface of revolution</u>.

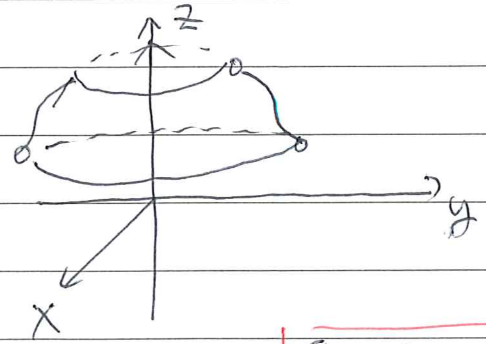
Let



$(x(t),y(t))$ be a piece of curves in x-y plane.

Rotate it around the y-axis to get a surface. Change notations

$$y\longmapsto z$$
$$x\longmapsto r$$



The surface can be described by the parametrization

$$\boxed{(\alpha,t)\longmapsto(r(t)\cos\alpha,\ r(t)\sin\alpha,\ z(t)),\quad \alpha\in[0,2\pi]}$$

$$\frac{\partial \vec{r}}{\partial \alpha} = (-r\sin\alpha,\ r\cos\alpha,\ 0)$$

$$\frac{\partial \vec{r}}{\partial t} = (r'\cos\alpha,\ r'\sin\alpha,\ z')$$

when $z' \neq 0$, $\frac{\partial \vec{r}}{\partial \alpha}$ and $\frac{\partial \vec{r}}{\partial t}$ are l. indept. when $z'=0$, but $r' \neq 0$, then, as $\det\begin{vmatrix} -r\sin\alpha & r\cos\alpha \\ r'\cos\alpha & r'\sin\alpha \end{vmatrix} = -rr' \neq 0$,

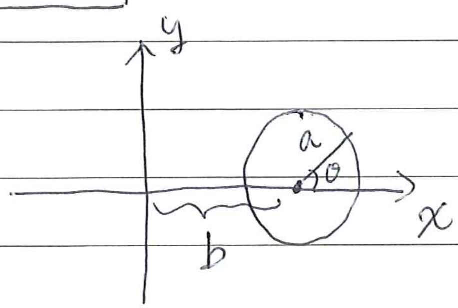
$\frac{\partial \vec{r}}{\partial \alpha}$ and $\frac{\partial \vec{r}}{\partial t}$ still indept. We conclude that as long as $(x(t), y(t))$ is regular curve, $\vec{r}(\alpha, t)$ is regular surface.

When a curve is given by explicit form $y = f(x)$ or implicit form $F(x,y) = c$, its corresponding surface of revolution are just

$$z = f(r),\quad r = (x^2+y^2)^{1/2}, \quad \text{or}$$

$$\boxed{F(r,z) = c}.$$

$\boxed{\text{e.g. } 2}$ Torus



$b > a > 0$

the curve

$$(x(\theta), y(\theta)) = (b + a\cos\theta,\ a\sin\theta)$$

the torus

$$(\alpha, \theta) \longmapsto ((b + a\cos\theta)\cos\alpha,\ (b+a\cos\theta)\sin\alpha,\ a\sin\theta)$$

$$\frac{\partial \vec{r}}{\partial \alpha} = (-(b+a\cos\theta)\sin\alpha,\ (b+a\cos\theta)\cos\alpha,\ 0)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-a\sin\theta\cos\alpha,\ -a\sin\theta\sin\alpha,\ a\cos\theta)$$

$$\frac{\partial \vec{r}}{\partial \alpha} \times \frac{\partial \vec{r}}{\partial \theta} = (a(b+a\cos\theta)\cos\theta\cos\alpha,\ (b+a\cos\theta)a\cos\theta\sin\alpha,\ a(b+a\cos\theta)\sin\theta)$$

$$\left|\frac{\partial \vec{r}}{\partial \alpha} \times \frac{\partial \vec{r}}{\partial \theta}\right| = a(b+a\cos\theta) > 0 \quad (\because b > a > 0)$$

So the torus is a regular surface.

the torus can be expressed as implicit form. Starting

with

$$(x-b)^2+y^2-a^2=0$$

$x \to r$, $y \to z$

$$(r-b)^2+z^2=a^2$$

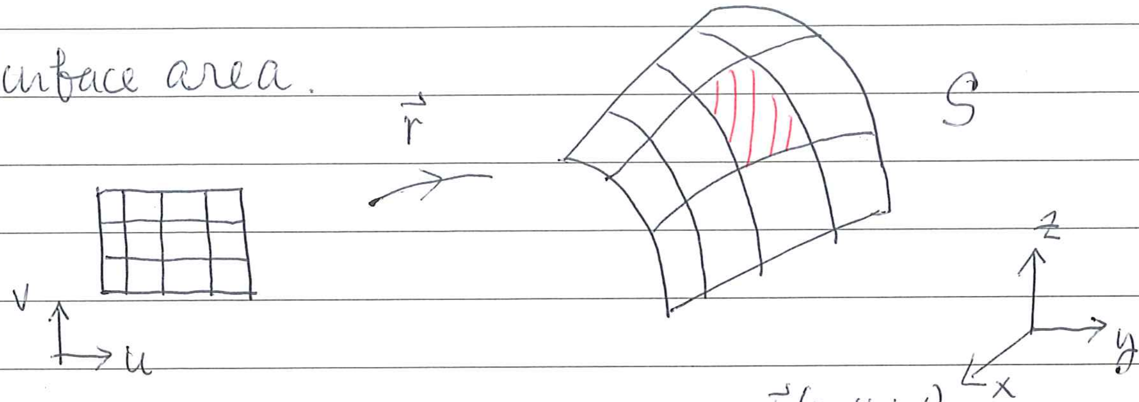
$$\sqrt{x^2+y^2} = \sqrt{a^2-z^2}+b$$

$$x^2+y^2 = a^2-z^2+2b\sqrt{a^2-z^2}+b^2$$

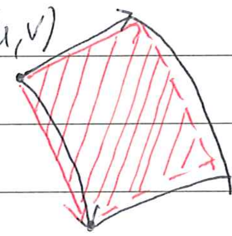
$\therefore (x^2+y^2+z^2-a^2-b^2)^2 = 4b(a^2-z^2)$ is the implict form of the torus.

Remark. The circle $x^2+y^2=b^2$ is a closed loop lying inside the torus which can't contract to a point inside the torus. So the solid torus V is not simply connected.

• Surface area.



A partition on $R = (u,v)$-plane introduces a generalized partition on the surface S.



area of (red) parallelogram at $\vec{r}(u,v)$, $\vec{r}(u+\Delta u,v)$, $\vec{r}(u,v+\Delta v)$, $\vec{r}(u+\Delta u,v+\Delta v)$

$\approx \frac{\partial \vec{r}}{\partial u}(u,v)\Delta u \times \frac{\partial \vec{r}}{\partial v}(u,v)\Delta v$

Consider a small portion vertices $\vec{r}(u,v)$, $\vec{r}(u+\Delta u,v)$, $\vec{r}(u,v+\Delta v)$, $\vec{r}(u+\Delta u,v+\Delta v)$.

$$\vec{r}(u+\Delta u,v) = \vec{r}(u,v) + \frac{\partial \vec{r}}{\partial u}(u,v)\,\Delta u + \cdots \text{higher order terms}$$

$$\sim \vec{r}(u,v) + \frac{\partial \vec{r}}{\partial u}(u,v)\,\Delta u$$

$$\vec{r}(u,v+\Delta v) = \vec{r}(u,v) + \frac{\partial \vec{r}}{\partial v}(u,v)\,\Delta v + \text{higher order term}$$

$$\sim \vec{r}(u,v) + \frac{\partial \vec{r}}{\partial v}(u,v)\,\Delta v.$$

So the area of this small portion is approximately

$$\left| \frac{\partial \vec{r}}{\partial u}(u,v)\,\Delta u \times \frac{\partial \vec{r}}{\partial v}(u,v)\,\Delta v \right|$$

$$= \left| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right| \Delta u\, \Delta v.$$

Summing up all this small portion and letting $\|P\| \to 0$, one get

$$\iint \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA(u,v).$$

It justifies the following definition: Let $\vec{r}: D \to S$ be a regular parametrization of the surface S. The surface area of S is defined to be

$$\boxed{\text{Area of } S = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA(u,v).}$$

$\boxed{\text{e.g. 3}}$ Find the surface area of the torus in e.g. 2.

We already have

$$|\vec{r}_\alpha \times \vec{r}_\theta| = a(b + a\cos\theta)$$

$\therefore$ the area of the torus is

$$\iint_R a(b + a\cos\theta)\, dA \qquad R = [0, 2\pi] \times [0, 2\pi]$$

$$= \int_0^{2\pi} \int_0^{2\pi} (ab + a^2\cos\theta)\, d\theta\, d\alpha$$

$$= 2\pi \int_0^{2\pi} (ab + a^2\cos\theta)\, d\theta$$

$$= 4\pi^2 ab. \quad \square$$

When the surface is in explicit form $z = f(x,y)$. The surface is given $(x, y, f(x,y))$

$$\frac{\partial \vec{r}}{\partial x} = (1, 0, f_x), \quad \frac{\partial \vec{r}}{\partial y} = (0, 1, f_y),$$

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = -f_x \hat{\imath} - f_y \hat{\jmath} + \hat{k}, \text{ so}$$

$$\left| \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} \right| = \sqrt{1 + f_x^2 + f_y^2}$$

$$= \sqrt{1 + |\nabla f|^2}.$$

Thus, we have,

When S is given by $z = f(x, y)$ over $D \subset \mathbb{R}^2$. then the area is

$$\boxed{\text{Area of } S = \iint_D \sqrt{1 + |\nabla f|^2} \, dA(x, y)}.$$

Finally, when S is given in implicit form

$S = \{(x, y, z) : F(x, y, z) = c, \nabla F(x, y, z) \neq 0\}$. We have some $z = f(x, y)$ s.t

$$F(x, y, f(x, y)) = c.$$

It follows that $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z} f_x = 0$, $\dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z} f_y = 0$, so

$$f_x = \frac{-F_x}{F_z}, \quad f_y = \frac{-F_y}{F_z}$$

$$\sqrt{1 + f_x^2 + f_y^2} = \frac{|\nabla F|}{|F_z|}.$$

We conclude, when S is given implicitly by $F(x, y, z) = c$

and it defines $z = f(x,y)$ over some D, then

$$\boxed{\text{area of } S = \iint_D \frac{|\nabla F|}{|F_z|}\, dA(x,y)}.$$

e.g. 4 Find the surface area of the paraboloid $z = x^2 + y^2$ bdd by $z = 4$.

$f(x,y) = x^2 + y^2$

$f_x = 2x,\ f_y = 2y \quad \therefore \sqrt{1 + f_x^2 + f_y^2} = \sqrt{1 + 4(x^2+y^2)}.$

The surface ~~an~~ is the paraboloid over D_2.

$$\therefore \text{Area} = \iint_{D_2} \sqrt{1 + 4(x^2+y^2)}\, dA(x,y)$$

$$= \int_0^{2\pi} \int_0^{2} \sqrt{1+4r^2}\, r\,dr\,d\theta$$

$$= \frac{2\pi}{2} \int_0^4 \sqrt{1+4t}\, dt$$

$$= \frac{\pi}{6}\left[(\sqrt{17})^3 - 1\right]. \ \#$$

- Surface Integral of a Function

Let G be a fcn on the surface S. Its integral is